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SOME PROPERTIES OF THE GENERALIZED SOLUTIONS OF ONE-DIMENSIONAL TWO-PHASE POROUS FLOW PROBLEMS*

A.V. DOMANSKII

In the context of the one-dimensional approximation, some qualitative results are obtained on the behaviour of the generalized solutions of the problem of the displacement of a wetting by a non-wetting fluid in a homogeneous porous medium, where there is monotonic dependence on the initial data, an over-all fluid consumption, and a pressure gradient. The asymptotic behaviour at long and short times is studied.

Assuming that the porous flow is linear or plane-parallel, we can write the system of equations of two-phase porous flow as

$$ms_t + x^{-n}(x^{-n}v_2)_x = 0, \quad -ms_t + x^{-n}(x^n v_1)_x = 0 \tag{1}$$
$$v_i = -k\mu_i^{-1} f_i(s)\, p_{ix}, \quad i = 1, 2; \; p_2 - p_1 = p_c(s)$$

Here, s is the saturation of the non-wetting phase, $0 \leqslant s \leqslant 1$, v_i, p_i are the porous flow velocity and pressure of the non-wetting $(i = 2)$ and wetting phases, m is the constant porosity, k is the permeability, μ_i is the phase velocity, f_i are the relative permeabilities, p_c is the capillary pressure, $dp_c/ds \geqslant 0$, $p_c(0) = 0$; $n = 0$ for linear flow, $n = 1$ for radial flow.

The initial-boundary conditions are

$$x = \nu, \; p_2 = p_1 = p_* = \text{const} \tag{2}$$
$$x = 1, \; v_1 = 0 \tag{3}$$
$$x = 1, \; p_2 = p_0 = \text{const}, \; \Delta p \equiv p_0 - p_* > 0 \tag{4}$$
$$x = 1, \; v_2 = -Q(t) \leqslant 0 \tag{5}$$
$$s(x, 0) = s^\circ(x), \; \nu \leqslant x \leqslant 1 \tag{6}$$

Here, $\nu = 0$ in the linear case, $\nu > 0$ is the pore radius for radical flow, and p_0, p_*, and Q are known quantities. Boundary condition (2) at the output section models the end-effect (/1/, p.367), whereby with simultaneous flow from the porous medium the pressures in the fluids equalize. Condition (3) means the absence of flow of displaced fluid at the input, while conditions (4) and (5) define the displacement mode, whereby the pressure or consumption of displaced fluid at the input section are given.

It was shown in /2/ that the one-dimensional problem of the displacement of the non-wetting by the wetting fluid is well-posed in the light of the end-effect; and the same was proved in /3/ for displacement of the wetting by the non-wetting fluid (see also, A.V. Domanskii, Porous flow of non-wetting fluids at the pore face, Dissertation, Novosibirsk, 1985).

We shall refer to (1)-(4), (6) as problem A, and to (1)-(3), (5), (6) as problem B.

In the same way as in /3/, system (1) can be transformed to a single equation for the saturation

$$s_t = (x^n a s_x + V(t)\, b)_x \tag{7}$$

while the boundary conditions can be written as

$$x = 1, \; x^n a s_x + Vb = 0, \; s(\nu^2, t) = 0 \tag{8}$$
$$a = kf_1 f_2 \chi\, dp_c/ds, \; b = \mu_2 f_1 \chi, \; \chi \equiv (f_2\mu_1 + f_1\mu_2)^{-1}$$

In the case of problem B the total flow velocity V is equal to $-2^n Q$, while for problem A it is given by the functional

$$V = -2^n (\Delta p - F(s(1, t)))/J(t) \tag{9}$$
$$F = \mu_2 \int_0^s \frac{a(\tau)}{kf_2(\tau)}\, d\tau, \quad J = \int_{\nu^2}^1 \rho(x, s)\, dx, \quad \rho = \mu_1\mu_2 x^{-n}\chi$$

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In future, we shall mean by the solutions of problems A and B the generalized solutions of problems (6)-(9) and (5)-(8) respectively.

In accordance with the physical meaning of the experimentally measured functions of capillary pressure and relative permeabilities

$$a(0) = a(1) = 0, \ a > 0, \ 0 < s < 1$$
$$b(0) = 1, \ b(1) = 0; \ -db/ds, \ dF/ds \geqslant 0 \tag{10}$$
$$0 < \alpha^{-1} \leqslant \rho(x, s) \leqslant \alpha = \text{const}$$

Thus, Eq.(7) degenerates for two values of the wanted solution.

For problem A we make the extra assumptions

$$s(1, t) = 1, \ t \geqslant 0, \ \Delta p > F(1), \ F(1) < \infty \tag{11}$$

We consider Eq.(7) in the domain $\Omega = \{\nu^2 < x < 1, \ 0 < t < T\}$, and G is the closure of Ω Definitions of the generalized solutions of problems A and B are given in /3/, and their existence was proved by the method of regularization. Since our assertions, proved below for the classical solutions of Eq.(7), remain in force, by a suitable passage to the limit with respect to the regularization parameter, for the generalized solutions, the regularization will be taken for granted in order to simplify the treatment.

Theorem 1. Let two distinct pressure gradients or total flows and initial saturation distributions be connected by the inequalities

$$\Delta p_1 \leqslant \Delta p_2, \ Q_1(t) \leqslant Q_2(t), \ s_1^\circ(x) \leqslant s_2^\circ(x), \ -\alpha \leqslant \partial p/\partial s \leqslant 0 \tag{12}$$

let condition (11) hold, and let s_j° $(j = 1, 2)$ be continuous monotonically non-decreasing functions. Then for the respective solutions $s_j(x, t)$ of problem A or B,

$$s_1(x, t) \leqslant s_2(x, t), \ (x, t) \in G.$$

The proof is obtained by applying the maximum principle to the function

$$w = \exp(-\beta t) \int_{s_1}^{s_2} a(\tau)\, d\tau, \qquad \beta = \text{const}$$

for which a suitable equation and boundary conditions are obtained. Here we use the fact that the function s /3/ is monotonically non-decreasing with respect to x. The last condition in (12) is used to find the sign of the difference $V_2 - V_1$.

Corollary 1. We consider the stationary solutions $s_0(x)$ of problems A and B, obtained with constant Δp_0 and Q_0. Let

$$Q(t) \leqslant Q_0, \ \Delta p \leqslant \Delta p_0, \ s^\circ(x) \leqslant s_0(x) \tag{13}$$

Then, by Theorem 1,

$$s(x, t) \leqslant s_0(x) \tag{14}$$

Theorem 2. If, in addition to the last condition in (12), $Q_t \geqslant 0, (a(s^\circ) s_x^\circ)_x \geqslant 0,$ then $s_t \geqslant 0$ in G.

The proof is obtained by applying, in the same way as in Theorem 1, the maximum principle to the function

$$H = \left(\int_0^s a(\xi)\, d\xi\right)_t$$

Corollary 2. In the conditions of theorems 1 and 2, we have a monotonic increase in the saturation function with respect to x and t. Then, it follows from the first two of Eqs.(1) that the porous flow rate of each phase is monotonic with respect to x, and from the second and third of Eqs.(1) and condition (3), that $v_1 \leqslant 0$ and $p_{1x} \geqslant 0$. Hence, noting the last equation of (1), we have $p_{2x} \geqslant 0$ and hence $v_2 \leqslant 0$

Lemma 1. Let

$$R(x) \in C^1[\nu^2, 1], \ |R_x| \leqslant \mu_0$$
$$0 \leqslant R \leqslant \delta, \quad \int_{\nu^2}^1 \eta R\, dx \leqslant \bar{\mu}, \quad \eta^{-1} \in L_q(\nu^2, 1), \quad 0 < q < 1$$

Then,

$$R \leqslant \max(N\bar{\mu}^r, \sqrt{N\mu_0\bar{\mu}^r}), \quad N = 2\left(\int_{\nu^2}^{1} \eta^{-q}\, dx\, \delta\right)^{1/(q+1)}, \quad r = \frac{q}{q+1}$$

Proof. Given any $\gamma, \gamma_0 \in [\nu^2, 1]$, by Hölder's inequality,

$$\left|\int_{\gamma}^{\gamma_0} R\, dx\right| \leqslant \int_{\nu^2}^{1} R\, dx \leqslant \left[\int_{\nu^2}^{1} \eta R\, dx\right]^{q/(q+1)} \left[\int_{\nu^2}^{1} \eta^{-q}\, dx \delta\right]^{1/(q+1)}$$

It remains to use proposition 1 of /4/. Let

$$0 \leqslant Q_0 - Q \in L_q(0, \infty), \; a^{-1}\partial\rho/\partial s \leqslant \mu_2^2 (\Delta p - F(1))^{-1} \tag{15}$$

Theorem 3. Let the conditions of Theorems 1 and 2, and (13), (15) hold, and let $\Delta p = \Delta p_0$. Then

$$0 \leqslant \max_x \int_{s}^{s_0} a(\tau)\, d\tau \leqslant Lt^{-\varkappa}, \quad 0 < \varkappa < \frac{1}{4}, \quad \varkappa, L = \mathrm{const} \tag{16}$$

Scheme of proof. Consider the case of problem $A, n = 0$. Subtracting from the stationary equation for s_0 the equation on s of type (7), we multiply the resulting equation by

$$\eta = \sin {}^1/_4\, \pi x + \cos {}^1/_4\, \pi x - 1 + {}^1/_2 x (2 - x) \tag{17}$$

We integrate this relation with respect to x, and then integrate by parts in the light of boundary conditions (8) for s, s_0. We obtain an equation from which, using (14) and the conditions on functions ρ, b of (12), (10), (15), and then integrating with respect to t, we have

$$\int_0^t y(\tau)\, d\tau \leqslant \frac{16}{\pi^2} \int_0^1 \eta(x)\, dx \equiv M, \quad 0 \leqslant y \equiv \int_0^1 \eta \int_s^{s_0} a(\xi)\, d\xi\, dx$$

It follows from the last inequality, since s is monotonic with respect to t, that $0 \leqslant y \leqslant Mt^{-1}$. Applying Lemma 1 (which is possible, since it was shown in /3/ that $a(s)s_x$, and $(s_0)s_{0x}$ are uniformly bounded with respect to x, t), we obtain (16) for sufficiently large t.

For radial flow, we can find η from (17) after replacing x by $\lambda = \ln(\nu^{-2}x)/\ln \nu^{-2}$.

In the case of problem B, it is sufficient to require that the first condition (15) holds and to use the functions $\eta = \sin {}^1/_2\pi x$, $n = 0$, $\eta = \sin {}^1/_2\pi\lambda$, $n = 1$

Note. The stationary solutions of problems A and B are $(n = 0)$

$$s_0 = \begin{cases} F_0^{-1}(x_1^{-1} x F_0(1)), & 0 \leqslant x \leqslant x_1 \\ 1, & x_1 \leqslant x \leqslant 1 \end{cases}$$

$$F_0(s) = \int_0^s \frac{a(\tau)}{b(\tau)}\, d\tau < \infty$$

where F_0^{-1} is the inverse of $F_0(s)$ and x_1 is given for problem A by

$$x_1 = (1 + (\Delta p - p_c(1))/F_0(1))^{-1}, \; \Delta p \geqslant p_c(1)$$

and in the case of problem B, by $x_1 = F_0(1)/Q_0, \; Q_0 \geqslant F(1)$.

If $\Delta p < p_c(1)$ or $Q_0 < F_0(1)$, then $s_0(1) < 1$.

These solutions are obtained in /5/ and describe the condition of capillary blocking of the displaced wetting fluid, the existence of which leads to incomplete extraction of hydrocarbon.

We define the function $\varphi(x, t)$ by the equation

$$\varphi = \int_0^{s(\varphi)} a(\tau)\, d\tau, \quad s(\varphi) = \begin{cases} 0, & \xi < 0 \\ F_0^{-1}(\mu\xi), & 0 \leqslant \xi \leqslant \xi_1 \\ 1, & \xi_1 \leqslant \xi \end{cases}$$

$$\xi = x - 1 + \varepsilon t, \; \mu = F_0(1)/\xi_1; \; \xi_1, \; \varepsilon = \mathrm{const} > 0.$$

Theorem 4. Let

$$\xi_1 \leqslant 1, \quad \mu < \min_{t,s}(-V) \equiv \mu^+, \quad 0 < \varepsilon \leqslant \min_s(-db/ds)(\mu^+ - \mu)$$

Then, for $t \geqslant t_1 = \xi_1 \varepsilon^{-1}$ we have condition (11), and for $t \geqslant t_2 = \varepsilon^{-1} \geqslant t_1$, the function $s(x, t) > 0,\ 0 < x \leqslant 1$.

Scheme of the proof. We introduce the function v by the equation

$$\int_0^t a(\tau)\, d\tau = \exp(\beta t)\, v + \varphi, \qquad \beta = \text{const}$$

In the domain $G_+ = \{1 - \varepsilon t \leqslant x \leqslant 1, 0 \leqslant t \leqslant t_2\}$ it will satisfy an equation and boundary condition from which, since $\varphi, s \geqslant 0$ and in G_+: $\varphi_{xx} = \varphi_{\xi\xi}$, $\varphi_x = \varphi_\xi$, $\varphi_t = \varepsilon\varphi_\xi$ (the subscript ξ means differentiation with respect to ξ) in the case of large β, we find by the maximum principle that $v \geqslant 0$ in G_+ Hence the theorem follows.

Corollary 3. Using Theorem 4 and the dissertation mentioned near the start, we can show that, for small t, the quantity $s(1, t)$ increases as $F_0^{-1}(t)$.

An example of relative permeability and capillary pressure functions which satisfy the conditions of Theorems 1-4 is given by

$$f_1(s) = (1 - s)(1 - s + {}^1/_3 s^2),\ f_2(s) = \mu^\circ(1 - f_1) + (1 - \mu^\circ)\, s^2 (3 - 2s)$$
$$p_c(s) = [s/(1,1 - s)]^{1/2},\ \mu^\circ = \mu_2/\mu_1 \leqslant 1$$

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THE SOLUTION OF PROBLEMS OF ELASTICITY THEORY BY THE METHOD OF ANALYTIC FUNCTIONS*

S.A. KULIYEV

Sherman's method /3/ is used to investigate some two-dimensional problems of elasticity theory for multiply connected regions. The solution is constructed by series expansion of the Kolosov-Muskhelishvili potential. Using Faber polynomials and conformal mapping, the original problem is reduced to the solution of linear systems of infinite algebraic equations in the expansion coefficients of analytical functions. The solution procedure is demonstrated by two examples.

Many problems of elasticity theory reduce /1-3/ to finding analytic functions of the complex variable $z = x + iy$ that are regular (or sinlge-valued) in a given region S and satisfy appropriate initial conditions.

For the plane problem (the first boundary-value problem), when S is bounded by several smooth closed contours $L_1, L_2, \ldots, L_k$ such that the last contour encloses all the previous

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